# Tutorial 1 (Abubakari Sumaila Salpawuni)

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## $Q_1$ – solution

By decomposition:

$$x_{lj} = \bar{x} + \underbrace{\bar{x}_l - \bar{x}}_{treatment} + \underbrace{x_{lj} - \bar{x}_l}_{residual}; \text{ where } \bar{x}_{..} = 30, \bar{x}_{1.} = 33, \bar{x}_{2.} = 29, \bar{x}_{3.} = 28$$

$$\begin{pmatrix} 34 & 24 & 36 & 39 & 32 \\ 30 & 31 & 34 & 23 & 27 \\ 23 & 28 & 28 & 30 & 31 \end{pmatrix} = \begin{pmatrix} 30 & 30 & 30 & 30 & 30 \\ 30 & 30 & 30 & 30 & 30 \\ 30 & 30 & 30 & 30 & 30 \end{pmatrix} + \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 & -1 \\ -2 & -2 & -2 & -2 & -2 \end{pmatrix} + \begin{pmatrix} 1 & -9 & 3 & 6 & -1 \\ 1 & 2 & 5 & -6 & -2 \\ -5 & 0 & 0 & 2 & 3 \end{pmatrix}$$

The *sum of squares* (SS) from the above are:

$$\begin{aligned} SS_{obs} &= SS_{mean} + SS_{treatment} + SS_{residual} \\ SS_{cor} &= SS_{obs} - SS_{mean} \end{aligned}$$

$$\begin{aligned} SS_{obs} &= 34^2 + 24^2 + \ldots + 30^2 + 31^2 = 13806 \\ SS_{mean} &= 30^2 + 30^2 + \ldots + 30^2 = 13500 \\ SS_{cor} &= 13806 - 13500 = 306 \\ SS_{trt} &= 3^2 + 3^2 + \ldots + (-2)^2 + (-2)^2 = 70 \\ SS_{res} &= 1^2 + (-9)^2 + \ldots + 2^2 + 3^2 = 236 \end{aligned}$$

| Source | DF | Sum of squares | Mean sum of squares | F-value |
|---|---|---|---|---|
| Treatment | 2 | 70 | 35 | |
| Error | 12 | 236 | 19.67 | 1.78* |
| Total | 14 | 306 | | |

Hypothesis formulation:

$$\begin{aligned} H_0 &: \mu_1 = \mu_2 = \mu_3 \\ H_1 &: \mu_1 \neq \mu_2 \neq \mu_3 \end{aligned}$$

$$\begin{aligned} F_{cal} &= 1.78 \\ F_{table} &= F_{[2,12,1-\alpha=0.95]} = 3.89 \end{aligned}$$

Since $F_{cal} < F_{table}$, we fail to reject $H_0$ and conclude that there is *no sufficient evidence* to conclude that temperature level appears to have an effect on the mean yield of the process.

## $Q_2$ – solution

$$f(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 \leq x \leq 1; 0 \leq y \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} f_X(x) &= \int_0^2 f(x,y)dy = \int_0^2 \frac{1}{3}(x+y)dy \\ &= \frac{1}{3}(xy + \frac{y^2}{2})\Big|_0^2 \\ &= \frac{2}{3}(x+1) \end{aligned}$$

$$\begin{aligned} E(X) &= \int_0^1 xf(x)dx = \int_0^1 x \cdot \frac{2}{3}(x+1)dx = \frac{5}{9} \\ E(X^2) &= \int_0^1 x^2 f(x)dx = \int_0^1 x^2 \cdot \frac{2}{3}(x+1)dx = \frac{7}{18} \\ Var(X) &= E(X^2) - (E(X))^2 = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_0^1 f(x,y)dx = \int_0^1 \frac{1}{3}(x+y)dx \\ &= \frac{1}{3}(\frac{x^2}{2} + xy)\Big|_0^1 \\ &= \frac{1}{6}(1+2y) \end{aligned}$$

$$\begin{aligned} E(Y) &= \int_0^1 yf(y)dy = \int_0^1 y \cdot \frac{1}{6}\left(\frac{x^2}{2} + xy\right)dy = \frac{11}{9} \\ E(Y^2) &= \int_0^1 y^2 f(y)dy = \int_0^1 y^2 \cdot \frac{1}{6}\left(\frac{x^2}{2} + xy\right)dy = \frac{16}{9} \\ Var(Y) &= E(Y^2) - (E(Y))^2 = \frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{23}{81} \end{aligned}$$

$$E(XY) = \int_0^1 \int_0^2 xyf(x,y)dydx = \int_0^1 \int_0^2 xy \cdot \frac{1}{3}(x+y)dydx$$
$$= \int_0^1 \int_0^2 \frac{1}{3}(x^2y + xy^2)dydx = \int_0^1 \frac{1}{3}\left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)\Bigg|_0^2 dx$$
$$= \int_0^1 \frac{1}{3}(2x^2 + \frac{8x}{3})dx = \frac{2}{3}$$

$$Cov(XY) = E(XY) - E(X)E(Y)$$
$$= \frac{2}{3} - \frac{5}{9} \cdot \frac{11}{9} = -\frac{1}{81}$$

$$\therefore \rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-1/81}{\sqrt{\frac{13}{162} \cdot \frac{23}{81}}} = -\mathbf{0.0818}$$

$$f(x,y) = \begin{cases} \frac{1}{22}(x+2y) & (1,1),(1,3),(2,1),(2,3) \\ \\ 0 & \text{elsewhere} \end{cases}$$

Hence, the joint bivariate distribution is:

| X/Y | 1 | 3 |
|---|---|---|
| 1 | 3/22 | 7/22 |
| 2 | 2/11 | 4/11 |

The marginals are:

$$f_X(x) = \begin{cases} \frac{5}{11} & \text{if } = 1 \\ \frac{6}{11} & \text{if } x = 2 \\ 0 & \text{elsewhere} \end{cases}$$
$$f_Y(Y) = \begin{cases} \frac{7}{22} & \text{if } y = 1 \\ \frac{15}{22} & \text{if } y = 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$E(X) = \sum xf(x) = (1) \cdot \frac{5}{11} + (2) \cdot \frac{6}{11} = \frac{17}{11}$$
$$E(X^2) = \sum x^2 f(x) = (1^2) \cdot \frac{5}{11} + (2^2) \cdot \frac{6}{11} = \frac{29}{11}$$
$$Var(X) = E(X^2) - (E(X))^2 = \frac{29}{11} - \frac{289}{121} = \frac{30}{121}$$

$$E(Y) = \sum y f(y) = (1) \cdot \frac{7}{22} + (3) \cdot \frac{15}{22} = \frac{52}{22}$$

$$E(Y^2) = \sum y^2 f(y) = (1^2) \cdot \frac{5}{11} + (9^2) \cdot \frac{15}{22} = \frac{142}{22}$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{142}{22} - \frac{676}{121} = \frac{105}{121}$$

$$E(X,Y) = \sum_x y f(x,y)$$

$$= (1)(1) \cdot \frac{3}{22} + (1)(3) \cdot \frac{7}{22} + (2)(1) \cdot \frac{2}{11} + (2)(3) \cdot \frac{4}{11} = \frac{40}{11}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{40}{11} - \frac{52}{22} \cdot \frac{17}{11} = -\frac{2}{121}$$

$$\therefore \rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-2/121}{\sqrt{\frac{30}{121} \cdot \frac{105}{121}}} = -\mathbf{0.0356}$$

## $Q_3$ – solution

The joint distribution table is shown below:

| RH1/RH2 | 18 | 24 | 30 |
|---|---|---|---|
| 24 | 0.09 | 0.15 | 0.06 |
| 30 | 0.18 | 0.30 | 0.12 |
| 36 | 0.03 | 0.05 | 0.02 |

The marginal distributions therefore are:

$$f_{RH1}(t_1) = \begin{cases} 0.30; & \text{if } t_1 = 24 \\ 0.60; & \text{if } t_1 = 30 \\ 0.10; & \text{if } t_1 = 36 \\ 0; & \text{elsewhere} \end{cases}$$

$$f_{RH2}(t_2) = \begin{cases} 0.30; & \text{if } t_2 = 18 \\ 0.50; & \text{if } t_2 = 24 \\ 0.20; & \text{if } t_2 = 30 \\ 0; & \text{elsewhere} \end{cases}$$

i.

$$\begin{aligned} P(t_1 = t_2) &= P(t_1 = 30, t_2 = 30) + P(t_1 = 24, t_2 = 24) \\ &= 0.15 + 0.12 = \mathbf{0.270} \end{aligned}$$

ii.

$$\begin{aligned} P(t_1 < 30, t_2 < 30) &= P(t_1 = 24, t_2 = 18) + P(t_1 = 24, t_2 = 24) \\ &= 0.09 + 0.15 = \mathbf{0.240} \end{aligned}$$

iii.

$$\begin{aligned}
P(t_1 > t_2) &= P(t_1 = 24, t_2 = 18) + P(t_1 = 30, t_2 = 18) + P(t_1 = 30, t_2 = 24) \\
&+ P(t_1 = 36, t_1 = 18) + P(t_1 = 36, t_2 = 24) + P(t_1 = 36, t_2 = 30) \\
&= 0.09 + 0.18 + 0.30 + 0.03 + 0.05 + 0.02 = \mathbf{0.670}
\end{aligned}$$

iv.

$$\begin{aligned}
E(X) &= \sum x f(x) \\
\therefore E(RH1) &= \sum t_1 f_{T_1}(t_1) = 24(0.30) + 30(0.60) + 36(0.10) = \mathbf{28.8} \\
\therefore E(RH2) &= \sum t_2 f_{T_1}(t_2) = 18(0.30) + 24(0.50) + 30(0.20) = \mathbf{23.40}
\end{aligned}$$

When ever the company wins a rehabilitation project categorized as **RH1**, the expected time to *completion* of the project is about 28.8 months. In the case of a project categorized as **RH2**, the time to completion is approximately 23.4months.